

# INTERACTION OF FLEXURAL PHONONS WITH ELECTRONS IN GRAPHENE: A GENERALIZED DIRAC EQUATION IN CORRUGATED SURFACES

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A generalized Dirac equation is derived in order to describe charge carriers moving in corrugated graphene, which is the case for temperatures above  $10^\circ K$  due to the presence of flexural phonons. Such interaction is taken into account by considering an induced metric, in the same spirit as the general relativity approach for the description of fermionic particle moving in a curved space-time. The resulting equation allows to include in a natural way the presence of other phonon branches as well as an external electromagnetic field. It also predicts non-linear effects which are not present in the usual vector potential approximation used in most of publications on the subject, as well as the possibility of controlling electronic conductivity using pure sinusoidal strain fields. The non-linear terms are important at high temperatures, and can also lead to interesting effects, like e.g. resonances between flexural phonons and external electromagnetic fields.

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## I. INTRODUCTION

Graphene is a new material that has been attracting a lot of attention since its experimental discovery [1]. This carbon allotrope has unique transport properties [2][3], like a high electronic mobility [4] and thermal conductivity [5], which are believed to be important for future applications in nano-devices [6][7]. However, there are certain discrepancies in the values of the electronic mobilities depending on whether the samples are suspended or in a substrate [8][9]. At low temperatures, impurity scattering can be responsible for this effect, which eventually leads to a metal-insulator transition since the mobility edge appears near the Fermi energy [10], as has been confirmed in graphene doped with H [11]. However, above  $T > 10^\circ K$  such discrepancies are believed to be a consequence of the crucial role of flexural phonon modes in the electron scattering, as has been shown very recently by applying tension to graphene sheets [9]. In fact, flexural modes are collective atomic displacements which are perpendicular to the graphene's plane [12], making it behave like a corrugated surface. From a microscopic point of view, the scattering results from changes in the distances between atoms, leading to fluctuations of  $\pi$ -orbitals electron wavefunctions overlaps [13].

However, since charge carriers in flat graphene are described by massless Dirac fermions [4, 14], it is natural to ask if it is possible to modify the Dirac equation taking into account the flexural mode interaction. Two paths can be followed in order to answer this question. One is to start from the usual tight-binding approach and use Taylor expansion of the overlap integral on the displacement field [12][13][15]. Here we present an alternative point of view, in which the interaction is included by making the observation that a graphene membrane can be considered as a curved space. The effective equation must be covariant due to simple and general physical arguments. The desired equation can be considered as akin to the Dirac equation in curved space-time. In fact, a similar approach has been taken in references [16][17][18] to study smooth ripples in graphene, using a metric that was considered asymptotically flat and with cylindrical symmetry. Our approach does not contain such requirements since the metric is general and thus can include a spatially non-decaying phonon field. This approach leads to interesting predictions, like the possibility of building gates for graphene electronic devices by superposing sinusoidal strain fields in different directions. Also, since the planar vibrational modes can be described by vector potential [12][13][15], our approach allows to include all phonon branches and the electromagnetic potential in a single equation. Note that in fact, the problem of the two dimensional Dirac equation including a vector potential has been solved recently [19][20]. As we shall show in the conclusion, the present approach has certain advantages over the Taylor expansion of the tight-binding parameters.

To finish this introduction, let us briefly sketch the ideas behind the present approach. For graphene

at low temperatures, its surface can be considered as flat. The corresponding unperturbed Hamiltonian operator used to describe charges evolving on the graphene sheet can be written as an effective Dirac equation [4, 14],

$$\hat{H} = -i\hbar v_F [\gamma^x \nabla_x + \gamma^y \nabla_y], \quad (1)$$

where  $v_F$  is the *Fermi velocity*, with

$$\gamma^x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (2)$$

Note that at this stage there is no difference between covariant and contravariant indices, because we have

$$\gamma^i \gamma^j + \gamma^j \gamma^i = \sigma_i \sigma_j + \sigma_j \sigma_i = 2 g_{ij} \mathbf{1}, \quad (i, j = x, y), \quad (3)$$

with  $g_{ij} = \delta_{ij}$ , so that obviously the contravariant metric raising the indices is also  $g^{ij} = \delta^{ij}$ . But this is no more true when the underlying two-dimensional space is not flat, but corrugated, with a non trivially deformed metric, as the case is for graphene at  $T \gg 10^\circ K$ . This is why a more careful treatment of this two-dimensional version of the Dirac equation should be considered. Luckily enough, the problem of covariant formulation of Dirac's equation in a curved space has been quite deeply investigated since a long time [21, 22], so that we can follow the same steps in this particular case: introduce the non-Euclidean two-dimensional metric on the graphene sheet, then adapt the Clifford algebra and find the Christoffel connection, and finally assemble all these in the covariant version of Dirac's equation.

The layout of the present article is the following: Section II describes the induced metric due to a corrugated surface, and in Section III we obtain the resulting covariant Dirac equation. Section IV deals with the application of the generalized Dirac equation to the description of electrons evolving on the corrugated graphene sheet, and finally, conclusions are given in Section V.

## II. THE INDUCED METRIC

At any finite temperature, phonons produce a displacement field ( $\vec{u}$ ). We start in this section by finding the resulting metrics and its Christoffel connection on corrugated graphene. Let the surface materialized by the graphene sheet be described by,

$$z = f(x, y, t), \quad (4)$$

where  $x, y$  are the coordinates in the graphene sheet, and  $z$  is the out of plane displacement. Notice that here we only consider the flexural phonon branch, since the planar phonons can be described by inclusion in a vectorial potential [12], whose solution is basically known [19][20].

As now the differential  $dz$  becomes a linear combination of  $dx$  and  $dy$ , we have

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy, \quad (5)$$

and the induced metric on the sheet is given by the following formula:

$$ds^2 = g_{ij} dx^i dx^j, \quad (i, j = x, y, z). \quad (6)$$

where  $g_{ij}$  is the metric tensor. Or, in a more explicit form,

$$ds^2 = dx^2 + dy^2 + dz^2 = dx^2 + dy^2 + \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)^2. \quad (7)$$

and it is obviously a metric in a 2-dimensional (but curved) space parametrized two space variables  $(x, y)$ . After opening the last expression we get the explicit form of the induced metric, which is

$$\tilde{g}_{jk} = \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{pmatrix} \quad (8)$$

This can be written as

$$\tilde{g}_{jk} = g_{jk} + h_{jk}, \quad \text{with } j, k = x, y. \quad (9)$$

where  $g_{ij} = \text{diag}(1, 1)$  is the flat metric in 2-dimensional space, and  $h_{ij}$  is the perturbation (supposed small as compared with 1) provoked by the corrugation of the sheet. The matrix (8) is symmetric and real, therefore it can be diagonalized by an appropriate linear transformation.

Also the inverse (contravariant) metric can be easily found. The determinant of the matrix corresponding to the covariant metric tensor (8) is easily found to be

$$\det \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{pmatrix} = (1 + (\partial_x f)^2 + (\partial_y f)^2); \quad (10)$$

and the inverse matrix, corresponding to the contravariant metric  $g^{jk}$  is

$$g^{jk} = \frac{1}{Q} \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial y}\right)^2 & -\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial x}\right)^2 \end{pmatrix} \quad (11)$$

where we used the abbreviate notation for the determinant,  $Q = 1 + (\partial_x f)^2 + (\partial_y f)^2$ .

The deformation  $h_{jk}$  is entirely composed of quadratic terms containing products of *spatial* partial derivatives of the deformation function  $f(x, y, t)$ ,

$$h_{jk} \simeq \partial_j f \partial_k f, \quad (12)$$

and it is easy to check that the corresponding Christoffel symbols reduce to

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2} \tilde{g}^{im} \left( \partial_j \tilde{g}_{mk} + \partial_k \tilde{g}_{jm} - \partial_m \tilde{g}_{jk} \right) = \tilde{g}^{im} \partial_m f \partial_{jk}^2 f. \quad (13)$$

Both quantities disappear when  $f = 0$ , and the metric becomes flat again.

The Dirac equation in two dimensions should be now generalized in order to incorporate the fact that the metric on the surface of constraint is no more flat, but curved. In the following section, we obtain the covariant generalization of the Dirac equation.

### III. THE COVARIANT DIRAC EQUATION

The equation we want to produce now can be written as the new deformed quantum-mechanical Hamiltonian acting on a two-component spinor  $\Psi$  as follows:

$$\hat{H} \Psi \sim \left[ \tilde{\gamma}^x \tilde{\nabla}_x + \tilde{\gamma}^y \tilde{\nabla}_y \right] \Psi. \quad (14)$$

Here not only the contravariant metric is deformed, but also the  $\gamma$ -matrices should be modified in order to satisfy new anti-commutation relations with the induced metric instead of the flat one as before; finally,  $\tilde{\nabla}_j$  contains not only the electromagnetic and on plane phonon interaction visualized by the

vector potential included in the usual gauge-invariant way, but also the Christoffel symbols of the metric  $\tilde{g}_{ij}$ :

$$\nabla_j \Psi = (\partial_j - e A_j) \Psi + \tilde{\Gamma}_{jk}^m \tilde{g}^{ki} \tilde{\Sigma}_{mk} \Psi, \quad (15)$$

where the Christoffel symbols  $\tilde{\Gamma}_{jk}^m$  are defined as usual, by means of the modified metric:

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2} \tilde{g}^{im} [\partial_j \tilde{g}_{mk} + \partial_k \tilde{g}_{jm} - \partial_m \tilde{g}_{jk}], \quad (16)$$

and  $\tilde{\Sigma}_{mk}$  is the matrix-valued anti-symmetric tensor defined by means of the modified gamma-matrices:

$$\tilde{\Sigma}_{mk} = \frac{1}{8} [\tilde{\gamma}_m \tilde{\gamma}_k - \tilde{\gamma}_k \tilde{\gamma}_m]. \quad (17)$$

This term is often called “spinorial connection”. [21, 22].

We are looking for two “deformed” generators of the Clifford algebra  $\tilde{\gamma}_x$  and  $\tilde{\gamma}_y$  that would satisfy

$$\tilde{\gamma}_x \tilde{\gamma}_x = -\left(\frac{\partial f}{\partial x}\right)^2 \mathbf{1}, \quad \tilde{\gamma}_x \tilde{\gamma}_y + \tilde{\gamma}_y \tilde{\gamma}_x = -2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \mathbf{1}, \quad \tilde{\gamma}_y \tilde{\gamma}_y = -\left(\frac{\partial f}{\partial y}\right)^2 \mathbf{1}. \quad (18)$$

In order to do this, we must introduce the third Pauli matrix, because the deformation of the sheet pushes it out of the strict two-dimensional plane  $(x, y)$ .

The undeformed Clifford algebra satisfying anti-commutation relations in a three-dimensional *flat* space is defined as follows:

$$\gamma_j \gamma_k + \gamma_k \gamma_j = 2g_{jk} \mathbf{1} \quad \text{with} \quad g_{jk} = \text{diag}(1, 1, 1), \quad j, k = 1, 2, 3, \quad (19)$$

is easily found to be generated by three Pauli matrices as follows:

$$\gamma_1 = \sigma_x, \quad \gamma_2 = \sigma_y, \quad \gamma_3 = \sigma_z, \quad (20)$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (21)$$

we have indeed

$$(\gamma_x)^2 = \mathbf{1}, \quad (\gamma_y)^2 = \mathbf{1}, \quad (\gamma_z)^2 = \mathbf{1}$$

and the three matrices anticommute with each other.

The ansatz for the two deformed space-like  $\gamma$ -matrices is simple: if we set

$$\tilde{\gamma}_x = \sigma_x + a\sigma_z, \quad \tilde{\gamma}_y = \sigma_y + b\sigma_z, \quad (22)$$

then the coefficients  $a$  and  $b$  should be, as it was easy to check,

$$a = \frac{\partial f}{\partial x}, \quad b = \frac{\partial f}{\partial y}, \quad (23)$$

so that

$$\tilde{\gamma}_x = \sigma_x + \frac{\partial f}{\partial x} \sigma_z, \quad \tilde{\gamma}_y = \sigma_y + \frac{\partial f}{\partial y} \sigma_z, \quad (24)$$

Now we have to produce their *contravariant* counterparts that appear in the Dirac equation [22]. We have:

$$\tilde{\gamma}^x = \tilde{g}^{xx} \tilde{\gamma}_x + \tilde{g}^{xy} \tilde{\gamma}_y, \quad \tilde{\gamma}^y = \tilde{g}^{yx} \tilde{\gamma}_x + \tilde{g}^{yy} \tilde{\gamma}_y, \quad (25)$$

which gives explicitly

$$\begin{aligned}\tilde{\gamma}^x &= \frac{1}{Q} \left[ \left( 1 + (\partial_y f)^2 \right) \sigma_x + (\partial_x f) \sigma_z - (\partial_x f)(\partial_y f) \sigma_y, \right], \\ \tilde{\gamma}^y &= \frac{1}{Q} \left[ \left( 1 + (\partial_x f)^2 \right) \sigma_y + (\partial_y f) \sigma_z - (\partial_x f)(\partial_y f) \sigma_x \right].\end{aligned}\quad (26)$$

Recalling that  $Q = 1 + (\partial_x f)^2 + (\partial_y f)^2$ , we can add and subtract in the numerators of the above formula respectively the following terms:  $(\partial_x f)^2 \sigma_x$  in the first one, and  $(\partial_y f)^2 \sigma_y$  in the second one; this will enable us to separate the undeformed matrices  $\sigma_x$  and  $\sigma_y$  and the genuine deformation terms containing spatial derivatives of  $f$ . This gives the following result:

$$\tilde{H} \sim \sigma^x \tilde{\nabla}_x + \sigma^y \tilde{\nabla}_y + \frac{\sigma_z}{Q} (\vec{\text{grad}} f \cdot \vec{\nabla}) - \frac{1}{Q} (\vec{\text{grad}} f \cdot \vec{\sigma}) (\vec{\text{grad}} f \cdot \vec{\nabla}) \quad (27)$$

where the vectors and their scalar products are two-dimensional, i.e. we mean:

$$\vec{\text{grad}} f = [\partial_x f, \partial_y f], \quad \vec{\sigma} = [\sigma_x, \sigma_y], \quad \vec{\nabla} = [\nabla_x, \nabla_y],$$

so that

$$(\vec{\text{grad}} f) \cdot (\vec{\sigma}) = \partial_x f \sigma_x + \partial_y f \sigma_y, \quad \text{etc.}$$

We see that already in the numerators we have not only linear, but also quadratic terms, notwithstanding the presence of quadratic terms in the denominator (contained in the normalizing factor  $Q$ ). If we decided to keep *linear terms only*, then the modified Hamiltonian would contain only one extra term proportional to the matrix  $\sigma_z$ :

$$\tilde{H}_{lin} \sim \hat{H}_0 + \delta \hat{H} = \sigma^x \nabla_x + \sigma^y \nabla_y + \sigma_z (\vec{\text{grad}} f \cdot \vec{\nabla}) \quad (28)$$

Note that also the differential operator  $\nabla$  is taken in its primary form, because the connection coefficients contain only quadratic expressions in derivatives of  $f$ . Observe that this equation is akin to the one obtained using a Taylor expansion of the overlap integral in a tight-binding approach.

However, if we choose to keep all terms up to quadratic ones, then we must take into account also the Christoffel coefficients in  $\tilde{\nabla}$ . It is easy to check the following explicit form of our Christoffel symbols; keeping only the second order expressions means that we can use the simplified formula in which  $\tilde{g}^{ij}$  is replaced by  $g^{ij} = \delta^{ij}$ . We have then:

$$\begin{aligned}\Gamma_{xx}^x &= \partial_x f \partial_{xx}^2 f, \quad \Gamma_{xy}^x = \Gamma_{yx}^x = \partial_x f \partial_{xy}^2 f, \quad \Gamma_{xx}^x = \partial_x f \partial_{yy}^2 f, \\ \Gamma_{xx}^y &= \partial_y f \partial_{xx}^2 f, \quad \Gamma_{xy}^y = \Gamma_{yx}^y = \partial_y f \partial_{xy}^2 f, \quad \Gamma_{yy}^y = \partial_y f \partial_{yy}^2 f,\end{aligned}\quad (29)$$

In covariant derivatives, these coefficients are contracted with  $g^{jk}$  and the *anti-symmetric* matrices  $\Sigma_{km}$ ,

$$\Gamma_{jk}^i g^{km} \Sigma_{im},$$

which, taking into account that only diagonal terms in  $g^{ik}$  do not vanish and are equal to one, leads to the following result when explicitated: for  $j = x$  we have

$$\Gamma_{xx}^x g^{xx} \Sigma_{xx} + \Gamma_{xy}^x g^{yy} \Sigma_{xy} + \Gamma_{xx}^y g^{xx} \Sigma_{yx} + \Gamma_{xy}^y g^{yy} \Sigma_{yy},$$

and for  $j = y$  we get a similar expression:

$$\Gamma_{yx}^x g^{xx} \Sigma_{xx} + \Gamma_{yy}^x g^{yy} \Sigma_{xy} + \Gamma_{yx}^y g^{xx} \Sigma_{yx} + \Gamma_{yy}^y g^{yy} \Sigma_{yy},$$

The non-vanishing metric tensor components are equal to one, whereas the  $\Sigma$ -matrices are anti-symmetric in their two lower indices, so what is left is only

$$\begin{aligned} \Gamma_{xy}^x \Sigma_{xy} + \Gamma_{xx}^y \Sigma_{yx} &= \left( \partial_x f \partial_{xy}^2 f - \partial_y f \partial_{xx}^2 f \right) \Sigma_{xy} = -(\partial_y f \partial_{xx}^2 f) \Sigma_{xy}, \\ \Gamma_{yy}^x \Sigma_{xy} + \Gamma_{yx}^y \Sigma_{yx} &= \left( \partial_y f \partial_{xx}^2 f - \partial_x f \partial_{yy}^2 f \right) \Sigma_{xy} = -(\partial_x f \partial_{yy}^2 f) \Sigma_{yx} \end{aligned} \quad (30)$$

These are the only terms remaining to be included in the covariant derivatives as follows:

$$\tilde{\nabla}_x = (\partial_x - eA_x) - (\partial_y f \partial_{xx}^2 f) \Sigma_{xy}, \quad \tilde{\nabla}_y = (\partial_y - eA_y) - (\partial_x f \partial_{yy}^2 f) \Sigma_{yx}.$$

In the Hamiltonian, they appear multiplied from the left by the corresponding  $\tilde{\gamma}$ -matrices, but here, evaluating the terms coming from the Christoffel connection coefficients, already quadratic in deformation  $f$ , we may keep only their undeformed version, which our case are just the two Pauli matrices  $\sigma^x$  and  $\sigma^y$ , so that the part of the deformed Hamiltonian keeping track of the Christoffel connection is

$$\begin{aligned} \sigma^x \tilde{\nabla}_x + \sigma^y \tilde{\nabla}_y &= \sigma^x (\partial_x - eA_x) + \sigma^x \left[ (\partial_x f) (\partial_{xy}^2 f) - (\partial_y f) (\partial_{xx}^2 f) \right] \Sigma_{xy}, \\ &+ \sigma^y (\partial_y - eA_y) + \sigma^y \left[ (\partial_y f) (\partial_{yx}^2 f) - (\partial_x f) (\partial_{yy}^2 f) \right] \Sigma_{yx}. \end{aligned} \quad (31)$$

Taking into account the fact that

$$(\sigma_x)^2 = \mathbf{1}, \quad (\sigma_y)^2 = \mathbf{1}, \quad \sigma_x \sigma_y = -\sigma_y \sigma_x, \quad \text{and} \quad \sigma^x \Sigma_{xy} = \frac{1}{4} \sigma_y,$$

and adding and subtracting terms like  $(\partial_y f) (\partial_{yy}^2 f)$  to the first expression and  $(\partial_x f) (\partial_{xx}^2 f)$  to the second, we get the following invariant form of the extra terms induced by the Christoffel connection:

$$\begin{aligned} &\frac{1}{4} \sigma^y \left[ (\partial_x f) (\partial_{xy}^2 f) + (\partial_y f) (\partial_{yy}^2 f) - (\partial_y f) (\partial_{yx}^2 f) - (\partial_y f) (\partial_{xx}^2 f) \right] \\ &+ \frac{1}{4} \sigma^x \left[ (\partial_y f) (\partial_{xy}^2 f) + (\partial_x f) (\partial_{xx}^2 f) - (\partial_x f) (\partial_{xx}^2 f) - (\partial_x f) (\partial_{yy}^2 f) \right] \end{aligned} \quad (32)$$

This in turn can be written in a more compact (and elegant !) way as follows:

$$\frac{1}{8} \vec{\sigma} \cdot \vec{\text{grad}} [(\partial_x f)^2 + (\partial_y f)^2] - \frac{1}{4} [\vec{\sigma} \cdot \vec{\text{grad}} f] \Delta f \quad (33)$$

with

$$\Delta f = \partial_{xx}^2 f + \partial_{yy}^2 f;$$

It is worthwhile to note that the expression (33) vanishes when  $f$  is a pure monochromatic wave,  $f = A \cos(\omega t - K_x x - K_y y)$ , but is different from zero as soon as there is a superposition of such expressions, e.g. for a standing wave. It is worthwhile to mention that this simple fact provides a nice

prediction of the present approach, since it means that one can control the electronic conductivity by mixing sinusoidal strain fields. For example, if two sinusoidal strains are applied in perpendicular directions using piezoelectric nano-devices, this will result in a very different electronic conductivity when compared with the case of only one sinusoidal deformation, due to the presence of the expression (33). The reason of such effect can be understood by observing that a pure sinusoidal deformation does not change distances inside the sheet, as can be easily verified by applying strain to a sheet of paper. If the strain is applied in only one direction, the sheet can be gently bended, while a second strain in a perpendicular direction produces real corrugation.

Now we are able to write down the full Hamiltonian for an electron on a sheet, taking into account that sheet's proper motions described by the deformation from horizontal plane given by  $z = f(x, y, t)$ , up to the second order (quadratic terms in derivatives of  $f$ ):

$$\begin{aligned} \hat{H} \sim \hat{H}_0 + \frac{1}{Q} \sigma_z (\vec{\text{grad}} f) \cdot (\vec{\nabla}) - \frac{1}{Q} (\vec{\text{grad}} f \cdot \vec{\sigma}) (\vec{\text{grad}} f \cdot \vec{\nabla}) \\ + \frac{1}{8} \vec{\sigma} \cdot \vec{\text{grad}} [(\partial_x f)^2 + (\partial_y f)^2] - \frac{1}{4} [\vec{\sigma} \cdot \vec{\text{grad}} f] \Delta f. \end{aligned} \quad (34)$$

The normalizing factor  $1/Q$  in front of two first contributions can be set to 1, because it contains the squares of derivatives of  $f$ , and if developed, will create terms of order 3 and 4 when multiplied by the terms behind.

The quantum-mechanical Hamiltonian is obtained from this expression by multiplying it by  $-\hbar$  and  $v_F$ . Let us define the operator of generalized momentum,

$$\hat{\vec{\pi}} = \hat{\vec{p}} - e\vec{A} = -i\hbar \vec{\text{grad}} - e\vec{A}$$

where  $\vec{A}$  is a vector potential that describes an electromagnetic field [19][20] or in plane longitudinal and transversal phonons [15]. Then we can write:

$$\begin{aligned} \hat{H} = v_F \begin{pmatrix} 0 & \hat{\pi}_x - i\hat{\pi}_y \\ \hat{\pi}_x + i\hat{\pi}_y & 0 \end{pmatrix} + v_F \begin{pmatrix} (\partial_x f) \hat{\pi}_x + (\partial_y f) \hat{\pi}_y & 0 \\ 0 & -(\partial_x f) \hat{\pi}_x - (\partial_y f) \hat{\pi}_y \end{pmatrix} \\ - i\hbar v_F \left[ (\vec{\text{grad}} f \cdot \vec{\sigma}) (\vec{\text{grad}} f \cdot \vec{\nabla}) + \frac{1}{8} \vec{\sigma} \cdot \vec{\text{grad}} [(\partial_x f)^2 + (\partial_y f)^2] - \frac{1}{4} [\vec{\sigma} \cdot \vec{\text{grad}} f] \Delta f \right]. \end{aligned} \quad (35)$$

This last equation is the generalized Dirac equation. For graphene, it is interesting to observe that in the Hamiltonian derived here, the terms  $\vec{\sigma} \cdot \vec{\text{grad}} [(\partial_x f)^2 + (\partial_y f)^2] / 8 - [\vec{\sigma} \cdot \vec{\text{grad}} f] \Delta f / 4$  are akin to the ones obtained from a vectorial potential approach [12]. However, Eq. (35) contains more terms, including a dominant linear correction in  $f$ .

#### IV. THE GENERALIZED DIRAC EQUATION ON CORRUGATED GRAPHENE

To show how Eq. (35) is used in the graphene case, we consider that the displacement field can be written using a simple set of basis functions provided by standing waves [23],

$$\vec{u} = \frac{2}{\sqrt{N}} \sum_{\mu, \mathbf{q} > 0} \vec{e}_\mu(\vec{q}) \left[ Q_{\mu, \vec{q}}^{(c)} \cos(\vec{q} \cdot \vec{R} - \omega_\mu(\vec{q})t) + Q_{\mu, \vec{q}}^{(s)} \sin(\vec{q} \cdot \vec{R} - \omega_\mu(\vec{q})t) \right] \quad (36)$$

where  $\mathbf{e}_\mu(\mathbf{q})$  is the polarization vector for a wave-vector  $\vec{q}$ ,  $\mu$  is the phononic branch,  $\omega_\mu(\vec{q})$  the dispersion relationship and  $\vec{R} = (x, y)$  is the position. Notice that the notation  $\mathbf{q} > 0$  indicates that

$q_x > 0$  and  $q_y > 0$ .  $Q_{\mu, \vec{q}}^{(c)}$  and  $Q_{\mu, \vec{q}}^{(s)}$  are operators given in terms of the creation (annihilation) phonon operators [23]  $a_{\mu, \vec{q}}$  ( $a_{\mu, \vec{q}}^\dagger$ ),

$$Q_{\mu, \vec{q}}^{(\alpha)} = \frac{1}{\sqrt{2M\omega_\mu(\vec{q})}} \left( a_{\mu, \vec{q}}^{(\alpha)} + a_{\mu, \vec{q}}^{(\alpha)\dagger} \right) \quad (37)$$

where  $\alpha$  runs over  $c$  and  $s$  and  $M$  is the carbon atom's mass.

Thus, for flexural phonons the function  $f(x, y, t)$  is given by,

$$f(x, y, t) = \frac{2}{\sqrt{N}} \sum_{\mathbf{q} > 0} \left[ Q_{F, \vec{q}}^{(c)} \cos \phi_{\mathbf{q}} + Q_{F, \vec{q}}^{(s)} \sin \phi_{\mathbf{q}} \right] \quad (38)$$

where  $\mu = F$  means that we are dealing with flexural phonons, and the phase  $\phi_{\mathbf{q}}$  is defined as  $\phi_{\mathbf{q}} = \vec{q} \cdot \vec{R} - \omega_\mu(\vec{q})t$ . For the flexural branch,  $\omega_F(\vec{q}) = \alpha_F \|\vec{q}\|^2$  where  $\alpha_F \approx 4.6 \times 10^{-7} \text{m}^2/\text{s}$ . Notice that  $Q_{F, \vec{q}}^{(c)}$  and  $Q_{F, \vec{q}}^{(s)}$  have length units, so  $f(x, y, t)$  has the same units, while  $\partial_x f$  and  $\partial_y f$  are dimensionless.

The generalized Dirac equation only needs as input the partial derivatives of  $f(x, y, t)$ ,

$$\partial_x f = \frac{2}{\sqrt{N}} \sum_{\mathbf{q} > 0} q_x \left[ Q_{F, \vec{q}}^{(c)} \cos \phi_{\mathbf{q}} + Q_{F, \vec{q}}^{(s)} \sin \phi_{\mathbf{q}} \right], \quad (39)$$

$$\partial_y f = \frac{2}{\sqrt{N}} \sum_{\mathbf{q} > 0} q_y \left[ Q_{F, \vec{q}}^{(c)} \cos \phi_{\mathbf{q}} + Q_{F, \vec{q}}^{(s)} \sin \phi_{\mathbf{q}} \right] \quad (40)$$

For example, if we keep only the linear correction in Eq. (35), the Hamiltonian reads as follows,

$$\begin{aligned} \tilde{H} = v_F \begin{pmatrix} 0 & \hat{\pi}_x - i\hat{\pi}_y \\ \hat{\pi}_x + i\hat{\pi}_y & 0 \end{pmatrix} + \frac{2v_F}{\sqrt{N}} \sum_{\mathbf{q} > 0} \left[ Q_{F, \vec{q}}^{(c)} \cos \phi_{\mathbf{q}} + Q_{F, \vec{q}}^{(s)} \sin \phi_{\mathbf{q}} \right] \times \\ \begin{pmatrix} q_x \hat{\pi}_x + q_y \hat{\pi}_y & 0 \\ 0 & -q_x \hat{\pi}_x - q_y \hat{\pi}_y \end{pmatrix}. \end{aligned} \quad (41)$$

Using the fact that the Fermi velocity is much higher than the flexural modes' speeds, we can drop the time dependence and apply perturbation theory to solve the equation, as will be shown in the forthcoming article.

The correction terms that results from  $\partial_x f$  and  $\partial_y f$  are of the order  $z_{\max}/\lambda$ , where  $z_{\max}/\lambda$  is the maximal vertical displacement of flexural mode, and  $\lambda$  is the wavelength of a typical ripple, usually around several atomic spacings  $a_0$ . Thus, the energy of flexural modes correction behaves as  $\sim v_F p_F (z_{\max}/a_0)$  in the linear approximation, the subsequent non-linear terms being proportional to the corresponding powers of the adimensional parameter  $z_{\max}/a_0$ , as obtained in other approaches [12].

## V. CONCLUSIONS

We have deduced a generalized Dirac equation that describes charge carriers moving in corrugated graphene. To produce such equation, an appropriate metric was found and the principle of covariance has been applied. The resulting equation contains a linear correction plus several non-linear terms. Some of these non-linear terms correspond to the Taylor expansion of the overlap integral approach[12], while others, including the linear correction, are new. Such terms could lead to interesting effects, like resonances between various phonon modes, or between flexural phonons and the oscillating external electromagnetic field. We expect these non-linear terms to become important at higher temperatures, since the dependence of the amplitude of  $f(x, y, t)$  at higher temperatures contains the square root of  $T$ , as can be easily shown by using the equipartition of energy principle. Also, we proposed construction of simple electronic gate by observing that a pure sinusoidal strain field has a very different effect on the electron dynamics than a superposition of sinusoidals propagating in different directions.



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